3 Sobolev Spaces

Exercise 3.1. Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{m,p}(\Omega)$ be a Cauchy sequence. Then for every $|\alpha| \leq m$, $\{D^\alpha u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$, thus there exists $g_\alpha \in L^p(\Omega)$ such that $D^\alpha u_n \to g_\alpha$ in $L^p(\Omega)$. Moreover, denoting by u the L^p-limit of u_n, then $D^\alpha u = g_\alpha$ in $\mathscr{D}'(\Omega)$, indeed

$$\langle D^\alpha u, \varphi \rangle = (-1)^{|\alpha|} \langle u, D^\alpha \varphi \rangle = (-1)^{|\alpha|} \lim_{n\to\infty} \langle u_n, D^\alpha \varphi \rangle = \lim_{n\to\infty} \langle D^\alpha u_n, \varphi \rangle = \langle g_\alpha, \varphi \rangle .$$

Exercise 3.2. The inclusion $W_0^{m,p}(\mathbb{R}^d) \subset W^{m,p}(\mathbb{R}^d)$ is trivial. In order to prove the other one, we need to show that $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{m,p}(\mathbb{R}^d)$. First we recall that $f_\varepsilon := f * \rho_\varepsilon \in C^\infty(\mathbb{R}^d)$ converge to f in $W^{m,p}(\mathbb{R}^d)$: indeed for every multi-index α with $|\alpha| \leq m$ it holds $D^\alpha f_\varepsilon = (D^\alpha f) * \rho_\varepsilon \to D^\alpha f$ in $L^p(\mathbb{R}^d)$. Given $\delta > 0$ let $\varepsilon > 0$ be such that $\|f - f_\varepsilon\|_{W^{m,p}(\mathbb{R}^d)} < \delta/2$.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be such that $\varphi \equiv 1$ on $B_1(0)$ and given $R > 0$, let $\varphi_R(x) := \varphi(x/R)$. We show that for R large enough, $\|f_\varepsilon - f_\varepsilon \varphi_R\|_{W^{m,p}(\mathbb{R}^d)} < \delta/2$ so that $\|f - f_\varepsilon\varphi_R\|_{W^{m,p}(\mathbb{R}^d)} < \delta$ and this concludes the proof.

For every multi-index α with $|\alpha| \leq m$, by the Leibniz rule (for smooth functions!), we have

$$D^\alpha(f_\varepsilon \varphi_R) = \sum_{\beta\leq\alpha} C_{\alpha,\beta} D^{\alpha-\beta} f_\varepsilon D^\beta \varphi_R = \sum_{\beta\leq\alpha} C_{\alpha,\beta} D^{\alpha-\beta} f_\varepsilon D^\beta \varphi \frac{1}{R^{|\beta|}}.$$

If $\beta = 0$, then by dominated convergence theorem, $D^\alpha f_\varepsilon \varphi_R \to D^\alpha f_\varepsilon$ in $L^p(\mathbb{R}^d)$ as $R \to \infty$. If $\beta \neq 0$, then

$$\|D^{\alpha-\beta} f_\varepsilon D^\beta \varphi \frac{1}{R^{|\beta|}}\|_{L^p} \leq \sup |D^\beta \varphi| \frac{1}{R^{|\beta|}} \|D^{\alpha-\beta} f_\varepsilon\|_{L^p(\mathbb{R}^d \setminus B_R)} \to 0 \quad \text{as } R \to \infty.$$

Since $C_{\alpha,0} = 1$, this proves that $D^\alpha(f_\varepsilon\varphi_R) \to D^\alpha f_\varepsilon$ in L^p as $R \to \infty$.

Exercise 3.3. Since $u \in L^\infty(\Omega)$ and Ω is a bounded domain, we have that $u \in L^p(\Omega)$ for every $p \in [1, \infty]$. Moreover, u is continuous on Ω. Let $\varphi : \Omega \to \mathbb{R}$ be a function of $\mathscr{D}(\Omega)$ so smooth with compact support in Ω and let us compute the partial derivatives of u in the sense of distributions

$$
\begin{aligned}
\langle\frac{\partial u}{\partial x},\varphi\rangle &= -\langle u,\frac{\partial\varphi}{\partial x}\rangle = -\int_\Omega u\frac{\partial\varphi}{\partial x}\\
&= -\int_0^1\int_{-1}^0 e^x\frac{\partial\varphi}{\partial x}(x,y)\,dxdy - \int_0^1\int_0^1(1+\sin(xy))\frac{\partial\varphi}{\partial x}(x,y)\,dxdy\\
&= -\int_0^1\varphi(0,y)\,dy + \int_0^1\int_{-1}^0 e^x\varphi(x,y)\,dxdy\\
&\quad+\int_0^1\varphi(0,y)\,dy + \int_0^1\int_0^1 y\cos(xy)\varphi(x,y)\,dxdy\\
&= \int_0^1\int_{-1}^0 e^x\varphi(x,y)\,dxdy + \int_0^1\int_0^1 y\cos(xy)\varphi(x,y)\,dxdy
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\frac{\partial u}{\partial y},\varphi\rangle &= -\langle u,\frac{\partial\varphi}{\partial y}\rangle = -\int_\Omega u\frac{\partial\varphi}{\partial y}\\
&= -\int_0^1\int_{-1}^0 e^x\frac{\partial\varphi}{\partial y}(x,y)\,dxdy - \int_0^1\int_0^1(1+\sin(xy))\frac{\partial\varphi}{\partial y}(x,y)\,dxdy\\
&= \int_0^1\int_0^1 x\cos(xy)\varphi(x,y)\,dxdy.
\end{aligned}
$$

Using that $|e^x|\leq 1$ in $\Omega\cap\{x<0\}$ and both $|x\cos(xy)|\leq 1$ and $|y\cos(xy)|\leq 1$ in $\Omega\cap\{x>0\}$, we deduce that the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ are in $L^p(\Omega)$ (and so $u\in W^{1,p}(\Omega)$) for every $1\leq p\leq\infty$.

Exercise 2.4. Let $\varphi\in\mathscr{D}(\Omega)$ and let $\widetilde{\Omega}\subset\subset\Omega$ such that $\mathrm{supp}\varphi\subset\widetilde{\Omega}$ and $\varepsilon<\operatorname{dist}(\widetilde{\Omega},\partial\Omega)$.

Consider the mollifications f_ε and g_ε, for $\varepsilon<\operatorname{dist}(\widetilde{\Omega},\partial\Omega)$ which are well-defined functions in $\widetilde{\Omega}$. If $p\in(1,+\infty)$, then $f_\varepsilon\to f$, $\partial_i f_\varepsilon\to\partial_i f$ in $L^p(\widetilde{\Omega})$ and $g_\varepsilon\to g$, $\partial_i g_\varepsilon\to\partial_i g$ in $L^{p'}(\widetilde{\Omega})$, from which we also deduce that

$$
\begin{gathered}
f_\varepsilon g_\varepsilon\to fg\\
\partial_i f_\varepsilon g_\varepsilon + f_\varepsilon\partial_i g_\varepsilon\to\partial_i fg + f\partial_i g,
\end{gathered}
$$

in $L^1(\Omega)$. Thus we can compute

$$
\begin{aligned}
\langle\partial_i(fg),\varphi\rangle &= -\int_{\widetilde{\Omega}} fg\partial_i\varphi = -\lim_{\varepsilon\to0}\int_{\widetilde{\Omega}} f_\varepsilon g_\varepsilon\partial_i\varphi\\
&= \lim_{\varepsilon\to0}\int_{\widetilde{\Omega}}(\partial_i f_\varepsilon g_\varepsilon + f_\varepsilon\partial_i g_\varepsilon)\varphi = \int_{\widetilde{\Omega}}(\partial_i fg + f\partial_i g)\varphi.
\end{aligned}
$$

If $p = 1$, then

$$\begin{aligned}\langle \partial_i(fg), \varphi\rangle &= -\int_{\widetilde{\Omega}} fg\partial_i\varphi = -\lim_{\varepsilon\to 0}\int_{\widetilde{\Omega}} f_\varepsilon g\partial_i\varphi \\ &= \lim_{\varepsilon\to 0}\int_{\widetilde{\Omega}}(\partial_i f_\varepsilon g + f_\varepsilon\partial_i g)\varphi = \int_{\widetilde{\Omega}}(\partial_i f g + f\partial_i g)\varphi,\end{aligned}$$

where, in the second equality we used that $f_\varepsilon \to f$ in $L^1(\widetilde{\Omega})$ and in the third equality we used the previous case : indeed $f_\varepsilon \in L^p(\widetilde{\Omega})$ for every $p \in (1, +\infty)$, being smooth, and $g \in L^\infty(\widetilde{\Omega})$ therefore also in $L^{p'}(\widetilde{\Omega})$. The case $p = \infty$ is analogous exchanging f and g.

Moreover, if $f, g \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, then $\partial_i(fg) = \partial_i f g + f\partial_i g \in L^2(\Omega)$, from which we conclude that $fg \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Exercise 2.5. We first show that $\widetilde{u}$ is well defined, namely that $-\infty < \widetilde{u}(x) < +\infty$ for all $x \in \mathbb{R}^n$. Let $\overline{x} \in \Omega$. We have $\widetilde{u}(x) < u(\overline{x}) + L|\overline{x} - x| < \infty$. Moreover since $u \in \mathrm{Lip}(\Omega)$, $|u(y) - u(\overline{x})| \leq L|y - \overline{x}|$ for every $y \in \Omega$, from which we deduce $u(y) \geq u(\overline{x}) - L|y - \overline{x}|$. Thus

$$u(\overline{x}) - L|x - \overline{x}| \leq u(y) + L|y - \overline{x}| - L|x - \overline{x}| \leq u(y) + L|y - x|,$$

and by taking the infimum all over $y \in \Omega$ we get

$$-\infty < u(\overline{x}) - L|x - \overline{x}| \leq \widetilde{u}(x).$$

Now we need to show that $\widetilde{u} \in \mathrm{Lip}(\mathbb{R}^n)$, with the same Lipschitz constant L of u. If $x, y \in \Omega$ then

$$|\widetilde{u}(x) - \widetilde{u}(y)| = |u(x) - u(y)| \leq L|x - y|.$$

If $x, y \in \Omega^c$ we have

$$|\widetilde{u}(x) - \widetilde{u}(y)| \leq \sup\left\{\Big|u(z) + L|x - z| - u(z) - L|y - z|\Big| \;:\; z \in \Omega\right\} \leq L|x - y|,$$

where we used the following inequality $|\inf f - \inf g| \leq \sup|f - g|$.

If now $x \in \Omega$ and $y \in \Omega^c$, we have

$$\widetilde{u}(y) - \widetilde{u}(x) = \widetilde{u}(y) - u(x) \leq u(x) + L|x - y| - u(x) = L|x - y|.$$

On the other hand, since $u \in \mathrm{Lip}(\Omega)$, $u(z) \geq u(x) - L|x - z|$ for all $z \in \Omega$. Thus

$$u(z) + L|z - y| \geq u(x) - L|z - x| + L|z - y| \geq u(x) - L|x - y|$$

and by taking the infimum all over $z \in \Omega$ we obtain

$$\widetilde{u}(y) \geq u(x) - L|x - y| = \widetilde{u}(x) - L|x - y|,$$

or, equivalently $\widetilde{u}(x) - \widetilde{u}(y) \leq L|x - y|$.